Work

Fact —

$$\text{Work Done} = \underbrace{\text{Force}}_{\text{component of force in the direction of travel}} \times \text{Distance}$$

What are the units of work?

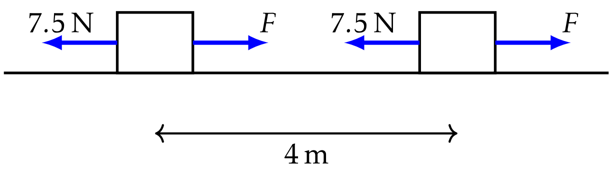
$$\begin{aligned} WD &= Force \times Distance \\ &= mass \times acceleration \times distance \end{aligned}$$

$$\begin{aligned} [WD] &= [mass] \times [acceleration] \times [distance] \\ &= M(LT^{-2})L \\ &= ML^2T^{-2} \end{aligned}$$

So our SI units will be $\mathrm{kgm^2s^{-2}}$. *There is a special name for this unit* J *(Joule) or* N m *(Newton metre).*

Example

A block of wood is pulled a distance of 4 m across a horizontal surface against resistances totalling 7.5 N. If the block moves at a constant velocity, find the work done against the resistance.



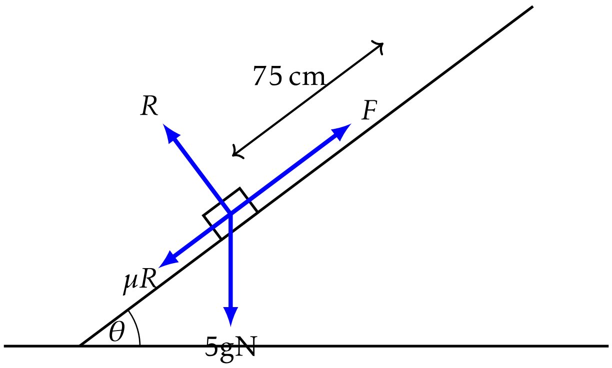
Since the block is moving with constant velocity, the acceleration is 0. *Therefore* $F = 7.5\,\mathrm{N}$. *In particular the work done against resistance is* $7.5 \times 4 = 30\,\mathrm{N\,m} = 30\,\mathrm{J}$

Example

A rough surface is inclined at $\tan^{-1}\frac{7}{24}$ to the horizontal. A body of mass 5 kg lies on the surface and is pulled at a uniform speed a distance of 75 cm up the surface by a force acting along a line of greatest slope. The coefficient of friction between the body and the surface is $\frac{5}{12}$. Find:

(a) the work done against gravity

(b) the work done against friction.



(a)

$$\begin{aligned}\text{Work done against gravity} &= \text{force} \times \text{vertical distance moved}\\ &= 5g \times 0.75 \cdot \sin\theta\\ &= 5g \times 0.75 \cdot \tfrac{7}{25}\\ &= 10.29\ldots\,\text{J}\\ &= 10.3\,\text{J}\end{aligned}$$

(b)

$$\begin{aligned}N2(\nwarrow): &\quad R - 5g\cos\theta = 0\\ \Rightarrow &\quad R = \frac{24}{5}g\\ \Rightarrow &\quad \mu R = \frac{5}{12}\frac{24}{5}g\\ &\quad = 2g\end{aligned}$$

Therefore work done against friction is $2g \times 0.75 = 14.7\,\text{J}$

Energy

What is energy?

A measure of the capacity to do work.
There are many types of energy:

- *kinetic energy*
- *gravitational potential energy*
- *elastic potential energy*
- *magnetic potential energy*
- *chemical potential energy*
- *nuclear potential energy*
- ...

Kinetic Energy

Example
Suppose a constant force F acts on a particle of mass m, over a distance s, accelerating the particle to a speed v. What is the work done on the body?

$$\begin{aligned} & v^2 - u^2 = 2as \\ \Rightarrow \quad & v^2 - 0^2 = 2as \\ & WD = Fs \\ & = mas \\ & = m \cdot \frac{v^2}{2} \\ & = \frac{1}{2}mv^2 \end{aligned}$$

Gravitational Potential Energy

Fact — The work done moving a body of mass m through a height h against gravity g is mgh. We can pick a height to be our 0 G.P.E. level and then measure gravitational potential energy from here.

Conservation of Energy

Why do we care about these different energies? Because of an extremely important result:

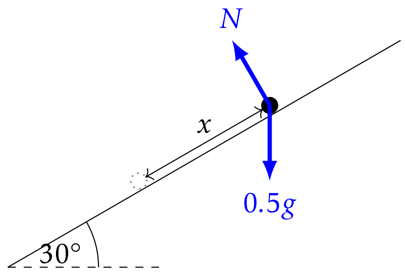
Fact (Principle of Conservation of Mechanical Energy) — Assuming gravity is the only external force which does work on the body, then the total energy possessed by the body is constant, ie:

$$\begin{aligned}\text{total energy} &= \underbrace{\text{kinetic energy}}_{K.E.} + \underbrace{\text{potential energy}}_{P.E.} \\ &= \text{constant}\end{aligned}$$

alternatively:

$$\text{Initial energy} = \text{Final Energy}$$

Example
A smooth plane is inclined at 30° to the horizontal. A particle of mass 0.5 kg slides down a line of greatest slope of the plane. The particle starts from rest at point A and passes point B with a speed $6\,\mathrm{m\,s^{-1}}$. Find the distance AB.



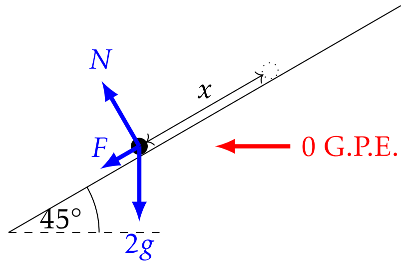
	K.E.	G.P.E.
Initial	0	0
Final	$\frac{1}{2}\cdot(0.5)\cdot 6^2$	$-0.5\cdot g\cdot x\sin 30°$

So by conservation of energy, $9 = \frac{x}{4}g \Rightarrow x = \frac{36}{g} = 3.67\,\mathrm{m}$

Fact (Work-Energy principle) — The change in the total energy of a particle is equal to the work done on the particle.

Example

A particle of mass 2 kg is projected with speed $8\,\mathrm{m\,s^{-1}}$ up a line of greatest slope of a rough plane inclined at 45° to the horizontal. The coefficient of friction between the particle and the plane is 0.4. Calculate the distance the particle travels up the plane before coming to instantaneous rest.

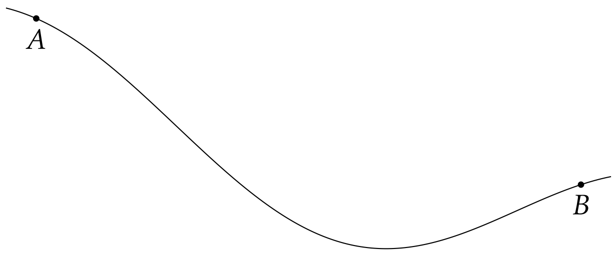


$$\Delta energy = (\frac{1}{2}mv^2 + mgh) - \frac{1}{2}mu^2$$
$$= 0 + 2 \cdot g \cdot x \sin 45° - \frac{1}{2} \cdot 2 \cdot 8^2$$
$$= \sqrt{2}gx - 64$$

$$\begin{aligned}
N2(\nwarrow): \quad & N - 2g\cos(45°) = 0 \\
\Rightarrow \quad & N = \sqrt{2}g \\
\textit{moving, so lim, eq:} \quad & F = \mu R \\
\Rightarrow \quad & F = 0.4 \cdot \sqrt{2}g \\
& \textit{work done against friction} = Fx \\
& = 0.4\sqrt{2}gx \\
\textit{work-energy principle}: \quad & \textit{work done against friction} = -\Delta energy \\
\Rightarrow \quad & 0.4\sqrt{2}gx = -\sqrt{2}gx + 64 \\
\Rightarrow \quad & x = \frac{64}{1.6\sqrt{2}g} \\
& = 3.298\ldots \\
& = 3.30\,\mathrm{m}
\end{aligned}$$

Example

A skier passes a point A on a ski-run moving downhill at $6\,\mathrm{ms^{-1}}$. After descending 50 m vertically the run begins to ascend. When the skier has ascended 25 m to a point B her speed is $4\,\mathrm{ms^{-1}}$. The skier and her skis have a combined mass of 55 kg. The total distance she travels from A to B is 1400 m. The non-gravitational resistance to motion are constant and have a total magnitude of 12 N. Calculate the work done by the skier.



$$\begin{aligned}\Delta KE &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2\\ &= \frac{1}{2}\cdot 55\cdot 4^2 - \frac{1}{2}\cdot 55\cdot 6^2\\ &= 550\end{aligned}$$

$$\begin{aligned}\Delta GPE &= mg\Delta h\\ &= 55\cdot 9.8\cdot(-25)\\ &= 13\,475\\ \textit{Total loss of energy} &= 550 + 13\,475 = 14\,025\end{aligned}$$

$$\begin{aligned}\textit{Total work done} &= -14\,025\\ \textit{Work done by skier} - \textit{work done against resistances} &= -14\,025\\ \textit{Work done by skier} &= \underbrace{12\cdot 1400}_{\textit{force}\times\textit{distance}} - 14\,025\\ &= 2775\end{aligned}$$

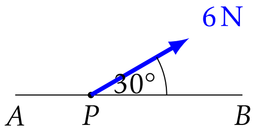
Therefore work done by skier is 2780 J (3 s.f.).

Work done by forces at angles

Example

The diagrams below show a constant force acting on a particle P. The force continues to act as the particle is made to move along a straight line path from A to B, a distance of 3 m. Find the work done by the force in each case.

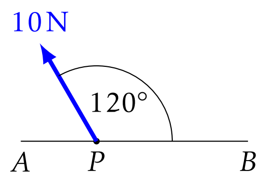
(a)



Component of force in direction of motion $6\cos(30°) = 3\sqrt{3}$N.

$$\begin{aligned} WD &= \textit{component of force} \times \textit{distance} \\ &= 3\sqrt{3} \times 3 \\ &= 9\sqrt{3}\text{J} \end{aligned}$$

(b)



Component of force in direction of motion $10\cos(120°) = -5$ N.

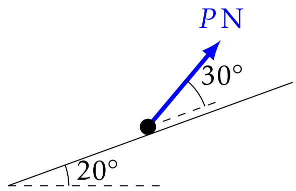
$$\begin{aligned} WD &= \textit{component of force} \times \textit{distance} \\ &= (-5) \times 3 \\ &= -15\,\text{J} \end{aligned}$$

Example (OCR M2 Jan 2007 Q4)

A skier of mass 80 kg is pulled up a slope which makes an angle of 20° with the horizontal. THe skier is to a constant frictional force of magnitude 70 N. The speed of the skier increases from 2 ms^{-1} at the point A to 5 ms^{-1} at the point B, and the distance AB is 25 m.

(i) By modelling the skier as a small object, calculate the work done by the pulling force as the skier moves from A to B.

(ii) It is given that the pulling force has constant magnitude PN, and that it acts at a constant angle of 30° above the slope (see diagram). Calculate P.



(i) Considering the energy of the skier initially we have:

$$\begin{aligned}
\textit{Initial Energy} &= \frac{1}{2}mv^2 + mgh \\
&= \frac{1}{2} \cdot 80 \cdot 2^2 + 0 \\
&= 160 \\
\textit{Final Energy} &= \frac{1}{2}mv^2 + mgh \\
&= \frac{1}{2} \cdot 80 \cdot 5^2 + 80 \cdot 9.8 \cdot 25 \sin(20°) \\
&= 7703.594 \cdots \text{J}
\end{aligned}$$

Therefore total work done is 7543.59... J*. However, this doesn't account for the work which needs to be done to overcome the resistive forces, ie* $70 \cdot 25 = 1750$ J*. Therefore the total work done by the pulling force is:* 9290 J *(3 s.f.).*

(ii)

$$\begin{aligned}
\textit{Work done} &= \textit{Component of force in direction of motion} \times \textit{distance} \\
9293.59\ldots &= P\cos(30°) \cdot 25 \\
\Rightarrow \qquad P &= \frac{9293.59\ldots}{25\cos(30°)} \\
&= 429.25\ldots \\
&= 429\,\text{N}
\end{aligned}$$

Power

Fact —

$$\begin{aligned}\text{Power} &= \frac{\text{Work done}}{\text{time}} \\ &= \text{the 'rate' of doing work} \\ &= \frac{d}{dt}(\text{Work done})\end{aligned}$$

If the force being applied is constant, then

$$\textit{Power} = \textit{Force} \times \textit{velocity}$$

What are the units of power?

$$\textit{Power} = \textit{Work Done} \div \textit{time}$$

$$\begin{aligned}[\textit{Power}] &= [\textit{WD}] \div [\textit{time}] \\ &= ML^2T^{-2} \div T \\ &= ML^2T^{-3}\end{aligned}$$

So our SI units will be $\text{kgm}^2\text{s}^{-3}$. *There is a special name for this unit* W *(Watt). Remember* 1000 W = 1 kW

Example

A lift, of total mass 1200 kg, rises a distance of 60 m in 20 s. What is the power output of the motor?

Work done to raise the block 60 m *is* $60 \times 1200 \times 9.8 = 705\,600$ J

$$\begin{aligned}\textit{Power} &= \frac{\textit{Work Done}}{\textit{time}} \\ &= \frac{705\,600}{20} \\ &= 35\,280\,\text{W}\end{aligned}$$

So the motor has an output of 35.3 kW.

Example

A van of mass 1250 kg is travelling along a horizontal road. The van's engine is working at 24 kW. The constant resistance to motion has a magnitude 600 N. Calculate

(a) the acceleration of the van when it is travelling at $6\,\text{ms}^{-1}$,

(b) the maximum speed of the van

Fact —

$$\text{Power} = \text{Force} \times \text{velocity}$$

(a) *Notice that* $P = Fv \Rightarrow F = \frac{P}{v} = \frac{24000}{6} = 4000\,\text{N}$.
There is a $600\,\text{N}$ *resistance, so the net force is* $3400\,\text{N}$. $F = ma \Rightarrow a = \frac{F}{m} = \frac{3400}{1250} = 2.72\,\text{ms}^{-2}$.

(b) *At the maximum speed the driving force will equal the resistance forces, ie* $\frac{P}{v} = 600 \Rightarrow v = \frac{P}{600} = \frac{24000}{600} = 40\,\text{ms}^{-1}$